

ON PROPERTIES OF SOLUTIONS OF QUASILINEAR SECOND-ORDER ELLIPTIC INEQUALITIES

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ABSTRACT. Let Ω be an unbounded open subset of \mathbb{R}^n , $n \geq 2$, and $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function such that

$$C_1|\zeta|^p \leq \zeta A(x, \zeta), \quad |A(x, \zeta)| \leq C_2|\zeta|^{p-1}$$

with some constants $C_1 > 0$, $C_2 > 0$, and $p > 1$ for almost all $x \in \Omega$ and for all $\zeta \in \mathbb{R}^n$. We obtain blow-up conditions and priori estimates for inequalities of the form

$$\operatorname{div} A(x, Du) + b(x)|Du|^\alpha \geq q(x)g(u) \quad \text{in } \Omega,$$

where $p - 1 \leq \alpha \leq p$ is a real number and, moreover, $b \in L_{\infty,loc}(\Omega)$, $q \in L_{\infty,loc}(\Omega)$, and $g \in C([0, \infty))$ are non-negative functions.

1. INTRODUCTION

Let Ω be an unbounded open subset of \mathbb{R}^n , $n \geq 2$, and $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function such that

$$C_1|\zeta|^p \leq \zeta A(x, \zeta), \quad |A(x, \zeta)| \leq C_2|\zeta|^{p-1}$$

with some constants $C_1 > 0$, $C_2 > 0$, and $p > 1$ for almost all $x \in \Omega$ and for all $\zeta \in \mathbb{R}^n$.

Denote $\Omega_{r_1, r_2} = \{x \in \Omega : r_1 < |x| < r_2\}$ and $B_{r_1, r_2} = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\}$, $0 < r_1 < r_2$. By B_r^z and S_r^z we mean the open ball and the sphere in \mathbb{R}^n of radius $r > 0$ and center at a point z . In the case of $z = 0$, we write B_r and S_r instead of B_r^0 and S_r^0 , respectively.

As in [14], the space $W_{p,loc}^1(\Omega)$ is the set of measurable functions belonging to $W_p^1(B_r \cap \Omega)$ for all real numbers $r > 0$ such that $B_r \cap \Omega \neq \emptyset$. The space $L_{p,loc}(\Omega)$ is defined analogously.

We consider inequalities of the form

$$\operatorname{div} A(x, Du) + b(x)|Du|^\alpha \geq q(x)g(u) \quad \text{in } \Omega, \quad (1.1)$$

where $D = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the gradient operator, $p - 1 \leq \alpha \leq p$ is a real number and, moreover, $b \in L_{\infty,loc}(\Omega)$, $q \in L_{\infty,loc}(\Omega)$, and $g \in C([0, \infty))$ are non-negative functions with $g(t) > 0$ for all $t > 0$.

A non-negative function $u \in W_{p,loc}^1(\Omega) \cap L_{\infty,loc}(\Omega)$ is called a solution of inequality (1.1) if $A(x, Du) \in L_{p/(p-1),loc}(\Omega)$ and

$$-\int_{\Omega} A(x, Du) D\varphi \, dx + \int_{\Omega} b(x)|Du|^\alpha \varphi \, dx \geq \int_{\Omega} q(x)g(u)\varphi \, dx$$

for any non-negative function $\varphi \in C_0^\infty(\Omega)$. In addition, we say that

$$u|_{\partial\Omega} = 0 \quad (1.2)$$

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if $\psi u \in \mathring{W}_p^1(\Omega)$ for any $\psi \in C_0^\infty(\mathbb{R}^n)$. Thus, in the case of $\Omega = \mathbb{R}^n$, condition (1.2) is valid for all $u \in W_{p,loc}^1(\mathbb{R}^n)$.

It is obvious that every solution of the equation

$$\operatorname{div} A(x, Du) + b(x)|Du|^\alpha = q(x)g(u) \quad \text{in } \Omega$$

is also a solution of inequality (1.1). Such equations and inequalities have traditionally attracted the attention of many mathematicians. They appear in the continuum mechanics, in particular, in the theory of non-Newtonian fluids and non-Newtonian filtrations [1, 20]. Other important examples arise in connection with the equations describing electromagnetic fields in spatially dispersive media [13] and the Matukuma and Batt-Faltenbacher-Horst equations that appear in the plasma physics [2, 3]. In so doing, of special interest is a phenomenon of the absence of non-trivial solutions which is known as the blow-up phenomenon.

Our aim is to obtain blow-up conditions and priori estimates for solutions of problem (1.1), (1.2). The questions treated below were investigated mainly for nonlinearities of the Emden-Fowler type $g(t) = t^\lambda$ [4, 6, 10, 16, 17, 19]. The case of general nonlinearity without lower-order derivatives was studied in [5, 8, 9, 18]. For inequalities containing lower-order derivatives, blow-up conditions were obtained in [7]. However, these results can not be applied to a class of inequalities, e.g., to the inequalities discussed in Examples 2.1–2.3.

Also, it should be noted that the authors of [5, 7, 8, 9, 18] use arguments based on the method of barrier functions. This method involves additional restrictions on the function A in the left-hand side of (1.1); therefore, it can not be applied to inequalities of the general form (1.1). For $\alpha = p - 1$, inequalities (1.1) were considered in [11, 12]. In the present paper, we managed to generalize results of [12] to the case of $p - 1 \leq \alpha \leq p$.

Throughout the paper, it is assumed that $S_r \cap \Omega \neq \emptyset$ for all $r > r_0$, where $r_0 > 0$ is some real number. For every solution of problem (1.1), (1.2) we denote

$$M(r; u) = \operatorname{ess\,sup}_{S_r \cap \Omega} u, \quad r > r_0,$$

where the restriction of u to $S_r \cap \Omega$ is understood in the sense of the trace and the $\operatorname{ess\,sup}$ on the right-hand side is with respect to $(n - 1)$ -dimensional Lebesgue measure on S_r .

We also put

$$\begin{aligned} g_\theta(t) &= \inf_{(t/\theta, \theta t)} g, \quad t > 0, \theta > 1, \\ q_\sigma(r) &= \operatorname{ess\,inf}_{\Omega_{r/\sigma, \sigma r}} q, \quad r > r_0, \sigma > 1 \end{aligned}$$

and

$$f_\sigma(r) = \frac{q_\sigma(r)}{1 + r q_\sigma^{(\alpha-p+1)/\alpha}(r) \operatorname{ess\,sup}_{\Omega_{r/\sigma, \sigma r}} b^{(p-1)/\alpha}}, \quad r > r_0, \sigma > 1. \quad (1.3)$$

2. MAIN RESULTS

Theorem 2.1. *Let*

$$\int_1^\infty (g_\theta(t)t)^{-1/p} dt < \infty, \quad (2.1)$$

$$\int_1^\infty g_\theta^{-1/\alpha}(t) dt < \infty, \quad (2.2)$$

and

$$\int_{r_0}^{\infty} (r f_{\sigma}(r))^{1/(p-1)} dr = \infty \quad (2.3)$$

for some real numbers $\theta > 1$ and $\sigma > 1$. Then any non-negative solution of (1.1), (1.2) is equal to zero almost everywhere in Ω .

Theorem 2.1 is proved in Section 3. Now, we demonstrate its exactness.

Example 2.1. Consider the inequality

$$\operatorname{div}(|Du|^{p-2}Du) + b(x)|Du|^{\alpha} \geq q(x)u^{\lambda} \quad \text{in } \mathbb{R}^n, \quad (2.4)$$

where $b \in L_{\infty,loc}(\mathbb{R}^n)$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ are non-negative functions such that

$$b(x) \leq b_0|x|^k, \quad b_0 = \text{const} > 0, \quad (2.5)$$

for almost all x in a neighborhood of infinity and

$$q(x) \sim |x|^l \quad \text{as } x \rightarrow \infty, \quad (2.6)$$

i.e. $k_1|x|^l \leq q(x) \leq k_2|x|^l$ with some constants $k_1 > 0$ and $k_2 > 0$ for almost all x in a neighborhood of infinity.

At first, let

$$\alpha + l(\alpha - p + 1) + k(p - 1) \leq 0 \quad (2.7)$$

(this condition implies that the second summand in the denominator on the right in (1.3) is bounded above by a constant for all $r > r_0$). According to Theorem 2.1, if

$$\lambda > \alpha \quad \text{and} \quad l \geq -p, \quad (2.8)$$

then any non-negative solution of (2.4) is equal to zero almost everywhere in \mathbb{R}^n . On the other hand, if $n \geq p$ and, moreover,

$$\lambda > \alpha \quad \text{and} \quad l < -p,$$

then

$$u(x) = \max\{|x|^{(l+p)/(p-\lambda-1)}, 1\}$$

is a solution of (2.4), where $b \equiv 0$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ is a non-negative function satisfying relation (2.6). This demonstrates the exactness of the second inequality in (2.8). The first inequality in (2.8) is also exact. Namely, in the case of $\lambda \leq \alpha$, it can be shown that (2.4) has a positive solution for all positive functions $b \in C(\mathbb{R})$ and $q \in C(\mathbb{R}^n)$.

Now, let

$$\alpha + l(\alpha - p + 1) + k(p - 1) > 0. \quad (2.9)$$

If

$$\lambda > \alpha \quad \text{and} \quad l \geq k - \alpha, \quad (2.10)$$

then in accordance with Theorem 2.1 any non-negative solution of (2.4) is equal to zero almost everywhere in \mathbb{R}^n . As we have said, the first inequality in (2.10) is exact. The second one is also exact. Really, in the case that

$$\lambda > \alpha \quad \text{and} \quad l < k - \alpha, \quad (2.11)$$

putting

$$u(x) = \max\{|x|^{(l-k+\alpha)/(\alpha-\lambda)}, \gamma\},$$

where $\gamma > 0$ is large enough, we obtain a solution of (2.4) with a non-negative function $b \in L_{\infty,loc}(\mathbb{R}^n)$ such that

$$b(x) \sim |x|^k \quad \text{as } x \rightarrow \infty \quad (2.12)$$

and a non-negative function $q \in L_{\infty,loc}(\mathbb{R}^n)$ satisfying relation (2.6).

Example 2.2. Let $b \in L_{\infty,loc}(\mathbb{R}^n)$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ be non-negative functions such that (2.5) holds and, moreover,

$$q(x) \sim |x|^l \log^m |x| \quad \text{as } x \rightarrow \infty. \quad (2.13)$$

At first, we assume that (2.7) is valid. By Theorem 2.1, the condition

$$\lambda > \alpha \quad \text{and} \quad l > -p$$

guarantees that any non-negative solution of (2.4) is equal to zero almost everywhere in \mathbb{R}^n for all $m \in \mathbb{R}$. We are interested in the case of the critical exponent $l = -p$. In this case, (2.7) can obviously be written as

$$k \leq \alpha - p.$$

In so doing, if

$$\lambda > \alpha \quad \text{and} \quad m \geq 1 - p, \quad (2.14)$$

then in accordance with Theorem 2.1 any non-negative solution of (2.4) is equal to zero almost everywhere in \mathbb{R}^n . As noted in Example 2.1, the first inequality in (2.14) is exact. At the same time, if $n > p$ and, moreover,

$$\lambda > \alpha \quad \text{and} \quad m < 1 - p,$$

then

$$u(x) = \max\{\log^{(m+p-1)/(p-\lambda-1)} |x|, \gamma\}$$

is a solution of (2.4) for enough large $\gamma > 0$, where $b \equiv 0$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ is a non-negative function satisfying relation (2.13). This demonstrates the exactness of the second inequality in (2.14).

Now, assume that (2.9) holds. According to Theorem 2.1, if

$$\lambda > \alpha \quad \text{and} \quad l > k - \alpha,$$

then any non-negative solution of (2.4) is equal to zero almost everywhere in \mathbb{R}^n for all $m \in \mathbb{R}$. Let us consider the case of the critical exponent $l = k - \alpha$. In this case relation (2.9) takes the form

$$k > \alpha - p. \quad (2.15)$$

By Theorem 2.1, the condition

$$\lambda > \alpha \quad \text{and} \quad m \geq -\alpha \quad (2.16)$$

implies that any non-negative solution of (2.4) is equal to zero almost everywhere in \mathbb{R}^n . The first inequality in (2.16) is exact. We show the exactness of the second inequality. Let

$$\lambda > \alpha \quad \text{and} \quad m < -\alpha. \quad (2.17)$$

Taking $\gamma > 0$ large enough, one can verify that

$$u(x) = \max\{\log^{(m+\alpha)/(\alpha-\lambda)} |x|, \gamma\}$$

is a solution of (2.4) for some non-negative functions $b \in L_{\infty,loc}(\mathbb{R}^n)$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ satisfying relations (2.12) and (2.13), respectively.

Example 2.3. Consider the inequality

$$\operatorname{div}(|Du|^{p-2}Du) + b(x)|Du|^\alpha \geq q(x)u^\alpha \log^\mu(1+u) \quad \text{in } \mathbb{R}^n, \quad (2.18)$$

where $b \in L_{\infty,loc}(\mathbb{R}^n)$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ are non-negative functions such that conditions (2.5) and (2.6) hold. We denote $\mu_* = \alpha$ for $\alpha > p-1$ and $\mu_* = p$ for $\alpha = p-1$.

Let (2.7) be valid. If

$$\mu > \mu_* \quad \text{and} \quad l \geq -p, \quad (2.19)$$

then in accordance with Theorem 2.1 any non-negative solution of (2.18) is equal to zero almost everywhere in \mathbb{R}^n . The first inequality in (2.19) is exact. Namely, if $\mu \leq \mu_*$, then (2.18) has a positive solution for all positive functions $b \in C(\mathbb{R}^n)$ and $q \in C(\mathbb{R}^n)$. In the case that $n \geq p$ and, moreover,

$$\mu > \mu_* \quad \text{and} \quad l < -p,$$

we can also specify a positive solution of (2.18), where $b \equiv 0$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ is a non-negative function satisfying relation (2.6). This solution is given by

$$u(x) = \max\{|x|^{(l+p)/(p-\alpha-1)} \log^{\mu/(p-\alpha-1)} |x|, \gamma\}$$

for $\alpha > p-1$ and

$$u(x) = e^{\max\{|x|^{(l+p)/(p-\mu)}, \gamma\}}$$

for $\alpha = p-1$, where $\gamma > 0$ is large enough. Hence, the second inequality in (2.19) is exact too.

Assume now that (2.9) is fulfilled. By Theorem 2.1, if

$$\mu > \mu_* \quad \text{and} \quad l \geq k - \alpha, \quad (2.20)$$

then any non-negative solution of (2.18) is equal to zero almost everywhere in \mathbb{R}^n . As we have previously said, the first inequality in (2.20) is exact. Let us show the exactness of the second inequality. Really, in the case that

$$\mu > \mu_* \quad \text{and} \quad l < k - \alpha,$$

putting

$$u(x) = e^{\max\{|x|^{(l-k+\alpha)/(\alpha-\mu)}, \gamma\}},$$

where $\gamma > 0$ is large enough, we obtain a solution of (2.18) with non-negative functions $b \in L_{\infty,loc}(\mathbb{R}^n)$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ satisfying relations (2.12) and (2.6), respectively.

Theorem 2.2. *Let there be real numbers $\theta > 1$ and $\sigma > 1$ such that (2.3) is valid and, moreover, at least one of conditions (2.1), (2.2) does not hold. If $u \not\equiv 0$ is a non-negative solution of (1.1), (1.2), then*

$$M(r; u) \geq F_\infty^{-1} \left(C \int_{r_0}^r (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \right) \quad (2.21)$$

for all sufficiently large r , where F_∞^{-1} is the function inverse to

$$F_\infty(\xi) = \left(\int_1^\xi (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} + \int_1^\xi g_\theta^{-1/\alpha}(t) dt$$

and the constant $C > 0$ depends only on $n, p, \theta, \sigma, \alpha, C_1$, and C_2 .

Theorem 2.3. *Let there be real numbers $\theta > 1$ and $\sigma > 1$ such that (2.1) and (2.2) are valid and, moreover, condition (2.3) does not hold. Then any non-negative solution of (1.1), (1.2) satisfies the estimate*

$$M(r; u) \leq F_0^{-1} \left(C \int_r^\infty (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \right) \quad (2.22)$$

for all sufficiently large r , where F_0^{-1} is the function inverse to

$$F_0(\xi) = \left(\int_\xi^\infty (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} + \int_\xi^\infty g_\theta^{-1/\alpha}(t) dt$$

and the constant $C > 0$ depends only on $n, p, \theta, \sigma, \alpha, C_1$, and C_2 .

Theorems 2.2 and 2.3 are proved in Section 3.

Remark 2.1. We assume by definition that $F_0(\infty) = 0$. Therefore, $F_0^{-1}(0) = \infty$ and (2.22) is fulfilled automatically if the integral in the right-hand side is equal to zero.

The following examples demonstrate an application of Theorems 2.2 and 2.3.

Example 2.4. Consider inequality (2.4), where $b \in L_{\infty,loc}(\mathbb{R}^n)$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ are non-negative functions such that relations (2.5), (2.6), and (2.9) are valid. By Theorem 2.2, if

$$0 \leq \lambda < \alpha \quad \text{and} \quad l > k - \alpha. \quad (2.23)$$

then any non-negative solution $u \not\equiv 0$ of (2.4) satisfies the estimate

$$M(r; u) \geq Cr^{(l-k+\alpha)/(\alpha-\lambda)}$$

for all r in a neighborhood of infinity, where the constant $C > 0$ does not depend on u .

Now, assume that condition (2.11) holds instead of (2.23). Then in accordance with Theorem 2.3 any non-negative solution of (2.4) satisfies the estimate

$$M(r; u) \leq Cr^{(l-k+\alpha)/(\alpha-\lambda)}$$

for all r in a neighborhood of infinity, where the constant $C > 0$ does not depend on u .

Example 2.5. Let $b \in L_{\infty,loc}(\mathbb{R}^n)$ and $q \in L_{\infty,loc}(\mathbb{R}^n)$ be non-negative functions such that (2.5) and (2.15) are valid and, moreover,

$$q(x) \sim |x|^{k-\alpha} \log^m |x| \quad \text{as } x \rightarrow \infty$$

In other words, we take the critical exponent $l = k - \alpha$ in formula (2.13). According to Theorem 2.2, if

$$0 \leq \lambda < \alpha \quad \text{and} \quad m > -\alpha, \quad (2.24)$$

then any non-negative solution $u \not\equiv 0$ of (2.4) satisfies the estimate

$$M(r; u) \geq C \log^{(m+\alpha)/(\alpha-\lambda)} r$$

for all r in a neighborhood of infinity, where the constant $C > 0$ does not depend on u .

In the case that (2.17) holds instead of (2.24), by Theorem 2.3, any non-negative solution of (2.4) satisfies the estimate

$$M(r; u) \leq C \log^{(m+\alpha)/(\alpha-\lambda)} r$$

for all r in a neighborhood of infinity, where the constant $C > 0$ does not depend on u .

It does not present any particular problem to verify that all estimates given in Examples 2.4 and 2.5 are exact.

3. PROOF OF THEOREMS 2.1–2.3

Throughout this section, we shall assume that $u \not\equiv 0$ is a non-negative solution of (1.1), (1.2). We need several preliminary assertions.

Lemma 3.1. *Let $M(r_1; u) = M(r_2; u) > 0$ for some real numbers $r_0 < r_1 < r_2$. Then*

$$\operatorname{ess\,inf}_{\Omega_{r_1, r_2}} q = 0.$$

Lemma 3.2. *Let $M(r_2; u) > M(r_1; u) \geq \beta M(r_2; u)$ and*

$$(M(r_2; u) - M(r_1; u))^{\alpha-p+1} (r_2 - r_1)^{p-\alpha} \operatorname{ess\,sup}_{\Omega_{r_1, r_2}} b \leq \frac{C_1}{4}$$

for some real numbers $r_0 < r_1 < r_2$ and $0 < \beta < 1$. Then

$$(M(r_2; u) - M(r_1; u))^{p-1} \geq C(r_2 - r_1)^p \operatorname{ess\,inf}_{\Omega_{r_1, r_2}} q \inf_{(\beta M(r_1; u), M(r_2; u))} g,$$

where the constant $C > 0$ depends only on n, p, α, β, C_1 , and C_2 .

Lemma 3.3. *Let $M(r_2; u) > M(r_1; u) \geq \beta M(r_2; u)$ and*

$$(M(r_2; u) - M(r_1; u))^{\alpha-p+1} (r_2 - r_1)^{p-\alpha} \operatorname{ess\,sup}_{\Omega_{r_1, r_2}} b \geq \lambda$$

for some real numbers $r_0 < r_1 < r_2$, $0 < \beta < 1$, and $\lambda > 0$. Then

$$(M(r_2; u) - M(r_1; u))^\alpha \operatorname{ess\,sup}_{\Omega_{r_1, r_2}} b \geq C(r_2 - r_1)^\alpha \operatorname{ess\,inf}_{\Omega_{r_1, r_2}} q \inf_{(\beta M(r_1; u), M(r_2; u))} g,$$

where the constant $C > 0$ depends only on $n, p, \alpha, \beta, \lambda, C_1$, and C_2 .

The proof of Lemmas 3.1–3.3 is given in Section 4.

We note that $M(\cdot; u)$ is a non-decreasing function on the interval (r_0, ∞) and, moreover, $M(r - 0; u) = M(r; u)$ for all $r > r_0$ (see Corollary 4.2, Section 4).

In the proof of Lemma 3.4, by c we denote various positive constants that can depend only on n, p, α, η, C_1 , and C_2 . In the proof of Lemma 3.5, analogous constants can also depend on τ , whereas in the proof of Lemma 3.6, they can depend only on $n, p, \alpha, \theta, \sigma, C_1$, and C_2 .

Lemma 3.4. *Let $0 < M(r_1 + 0; u) \leq \eta^{-1/2} M(r_2; u)$ and $r_2 \leq \tau r_1$ for some real numbers $r_0 \leq r_1 < r_2$, $\eta > 1$, and $\tau > 1$. Then at least one of the following two inequalities is valid:*

$$\int_{M(r_1+0; u)}^{M(r_2; u)} (g_\eta(t)t)^{-1/p} dt \geq C \int_{r_1}^{r_2} q_\tau^{1/p}(\xi) d\xi, \quad (3.1)$$

$$\int_{M(r_1+0; u)}^{M(r_2; u)} g_\eta^{-1/\alpha}(t) dt \geq C \int_{r_1}^{r_2} (\xi f_\tau(\xi))^{1/(p-1)} d\xi, \quad (3.2)$$

where the constant $C > 0$ depends only on n, p, α, η, C_1 , and C_2 .

Proof. Consider a finite sequence of real numbers $\rho_k < \dots < \rho_1 < \rho_0$ defined as follows. We take $\rho_0 = r_2$. Assume that ρ_i is already known. In the case of $\rho_i = r_1$, we take $k = i$ and stop; otherwise we put

$$\rho_{i+1} = \inf\{\xi \in (r_1, \rho_i) : M(\xi; u) > \eta^{-1/2} M(\rho_i; u)\}.$$

It can easily be seen that this procedure must terminate at a finite step. In so doing, we have

$$\eta^{-1/2} M(\rho_i; u) \leq M(\rho_{i+1} + 0; u), \quad i = 0, \dots, k-1 \quad (3.3)$$

and

$$M(\rho_{i+1}; u) \leq \eta^{-1/2} M(\rho_i; u), \quad i = 0, \dots, k-2. \quad (3.4)$$

Let Ξ_1 be the set of integers $i \in \{0, \dots, k-1\}$ such that $M(\rho_i; u) > M(\rho_{i+1} + 0; u)$ and

$$(M(\rho_i; u) - M(\rho_{i+1} + 0; u))^{\alpha-p+1} (\rho_i - \rho_{i+1})^{p-\alpha} \operatorname{ess\,sup}_{\Omega_{\rho_{i+1}, \rho_i}} b \leq \frac{C_1}{4}. \quad (3.5)$$

Also let $\Xi_2 = \{0, \dots, k-1\} \setminus \Xi_1$. By Lemma 3.2,

$$\begin{aligned} (M(\rho_i; u) - M(\rho_{i+1} + 0; u))^{p-1} &\geq c(\rho_i - \rho_{i+1})^p \operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q \\ &\quad \times \inf_{(\eta^{-1/2} M(\rho_{i+1} + 0; u), M(\rho_i; u))} g \end{aligned} \quad (3.6)$$

for all $i \in \Xi_1$. In turn, Lemmas 3.1 and 3.3 imply the inequality

$$\begin{aligned} (M(\rho_i; u) - M(\rho_{i+1} + 0; u))^\alpha \operatorname{ess\,sup}_{\Omega_{\rho_{i+1}, \rho_i}} b &\geq c(\rho_i - \rho_{i+1})^\alpha \operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q \\ &\quad \times \inf_{(\eta^{-1/2} M(\rho_{i+1} + 0; u), M(\rho_i; u))} g \end{aligned} \quad (3.7)$$

for all $i \in \Xi_2$.

Let us show that

$$\int_{M(\rho_k + 0; u)}^{M(\rho_0; u)} (g_\eta(t)t)^{-1/p} dt \geq c \operatorname{ess\,inf}_{\Omega_{r_1, r_2}} q^{1/p} \sum_{i \in \Xi_1} (\rho_i - \rho_{i+1}). \quad (3.8)$$

Really, taking into account (3.3), we have

$$\begin{aligned} \int_{\eta^{-1/2} M(\rho_i; u)}^{M(\rho_i; u)} (g_\eta(t)t)^{-1/p} dt &\geq c(M(\rho_i; u) - M(\rho_{i+1} + 0; u))^{(p-1)/p} \\ &\quad \times \sup_{(\eta^{-1/2} M(\rho_{i+1} + 0; u), M(\rho_i; u))} g^{-1/p} \end{aligned}$$

for all $i \in \Xi_1$. Combining this with the inequality

$$\begin{aligned} (M(\rho_i; u) - M(\rho_{i+1} + 0; u))^{(p-1)/p} &\quad \sup_{(\eta^{-1/2} M(\rho_{i+1} + 0; u), M(\rho_i; u))} g^{-1/p} \\ &\geq c(\rho_i - \rho_{i+1}) \operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q^{1/p} \end{aligned} \quad (3.9)$$

which is a consequence of (3.6), we immediately obtain

$$\int_{\eta^{-1/2} M(\rho_i; u)}^{M(\rho_i; u)} (g_\eta(t)t)^{-1/p} dt \geq c(\rho_i - \rho_{i+1}) \operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q^{1/p}$$

for all $i \in \Xi_1$. By (3.4), for different $i \in \{0, \dots, k-2\}$, the domains of integration for the integrals in the left-hand side of the last estimate intersect in a set of zero

measure. Thus, summing this estimate over all $i \in \Xi_1 \cap \{0, \dots, k-2\}$, one can conclude that

$$\int_{M(\rho_{k-1};u)}^{M(\rho_0;u)} (g_\eta(t)t)^{-1/p} dt \geq c \operatorname{ess\,inf}_{\Omega_{r_1,r_2}} q^{1/p} \sum_{i \in \Xi_1 \cap \{0, \dots, k-2\}} (\rho_i - \rho_{i+1}). \quad (3.10)$$

If $k-1 \notin \Xi_1$, then (3.10) obviously implies (3.8). Consider the case of $k-1 \in \Xi_1$. From (3.3), it follows that

$$\begin{aligned} \int_{M(\rho_k+0;u)}^{\eta^{1/2}M(\rho_k+0;u)} (g_\eta(t)t)^{-1/p} dt &\geq c(M(\rho_{k-1};u) - M(\rho_k+0;u))^{(p-1)/p} \\ &\quad \times \sup_{(\eta^{-1/2}M(\rho_k+0;u), M(\rho_{k-1};u))} g^{-1/p}. \end{aligned}$$

Combining this with inequality (3.9), where $i = k-1$, we have

$$\int_{M(\rho_k+0;u)}^{\eta^{1/2}M(\rho_k+0;u)} (g_\eta(t)t)^{-1/p} dt \geq c(\rho_{k-1} - \rho_k) \operatorname{ess\,inf}_{\Omega_{\rho_k, \rho_{k-1}}} q^{1/p}.$$

Finally, summing the last expression and (3.10), we obtain (3.8).

Further, let us show that

$$\int_{M(\rho_k+0;u)}^{M(\rho_0;u)} g_\eta^{-1/\alpha}(t) dt \geq c \sup_{r \in (r_1, r_2)} (rf_\tau(r))^{1/(p-1)} \sum_{i \in \Xi_2} (\rho_i - \rho_{i+1}). \quad (3.11)$$

Taking into account (3.7), we have

$$\begin{aligned} &(M(\rho_i;u) - M(\rho_{i+1}+0;u)) \sup_{(\eta^{-1/2}M(\rho_{i+1}+0;u), M(\rho_i;u))} g^{-1/\alpha} \\ &\geq c(\rho_i - \rho_{i+1}) \sup_{r \in (r_1, r_2)} (rf_\tau(r))^{1/(p-1)} \end{aligned} \quad (3.12)$$

for all $i \in \Xi_2$. Really, if

$$\operatorname{ess\,sup}_{\Omega_{\rho_{i+1}, \rho_i}} b = 0,$$

then

$$\operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q = 0$$

by (3.7). Hence, $f_\tau(r) = 0$ for all $r \in (\rho_{i+1}, \rho_i)$ and (3.12) is evident. Now, let

$$\operatorname{ess\,sup}_{\Omega_{\rho_{i+1}, \rho_i}} b > 0,$$

then (3.7) implies the estimate

$$\begin{aligned} &(M(\rho_i;u) - M(\rho_{i+1}+0;u)) \sup_{(\eta^{-1/2}M(\rho_{i+1}+0;u), M(\rho_i;u))} g^{-1/\alpha} \\ &\geq c(\rho_i - \rho_{i+1}) \left(\frac{\operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q}{\operatorname{ess\,sup}_{\Omega_{\rho_{i+1}, \rho_i}} b} \right)^{1/\alpha} \end{aligned}$$

and, to establish the validity of (3.12), it remains to note that

$$\frac{\operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q}{\operatorname{ess\,sup}_{\Omega_{\rho_{i+1}, \rho_i}} b} \geq \sup_{r \in (r_1, r_2)} (rf_\tau(r))^{\alpha/(p-1)}.$$

In turn, combining (3.12) with the inequality

$$\begin{aligned} \int_{M(\rho_{i+1}+0;u)}^{M(\rho_i;u)} g_\eta^{-1/\alpha}(t) dt &\geq (M(\rho_i;u) - M(\rho_{i+1}+0;u)) \\ &\quad \times \sup_{(\eta^{-1/2}M(\rho_{i+1}+0;u), M(\rho_i;u))} g^{-1/\alpha} \end{aligned}$$

which follows from (3.3), we have

$$\int_{M(\rho_{i+1}+0;u)}^{M(\rho_i;u)} g_\eta^{-1/\alpha}(t) dt \geq c(\rho_i - \rho_{i+1}) \sup_{r \in (r_1, r_2)} (rf_\tau(r))^{1/(p-1)} \quad (3.13)$$

for all $i \in \Xi_2$. The last inequality obviously implies (3.11).

In the case of

$$\sum_{i \in \Xi_1} (\rho_i - \rho_{i+1}) \geq \frac{r_2 - r_1}{2}, \quad (3.14)$$

estimate (3.8) allows us to establish the validity of (3.1). On the other hand, if (3.14) is not valid, then

$$\sum_{i \in \Xi_2} (\rho_i - \rho_{i+1}) \geq \frac{r_2 - r_1}{2}$$

and, using (3.11), we immediately obtain (3.2).

The proof is completed. \square

Lemma 3.5. *Let $0 < M(r_2; u) \leq \eta^{1/2}M(r_1 + 0; u)$ and $\tau^{1/2}r_1 \leq r_2$ for some real numbers $r_0 \leq r_1 < r_2$, $\eta > 1$, and $\tau > 1$. Then either estimate (3.2) holds or*

$$\int_{M(r_1+0;u)}^{M(r_2;u)} g_\eta^{-1/(p-1)}(t) dt \geq C \int_{r_1}^{r_2} (\xi f_\tau(\xi))^{1/(p-1)} d\xi, \quad (3.15)$$

where the constant $C > 0$ depends only on $n, p, \alpha, \eta, \tau, C_1$, and C_2 .

Proof. We take the maximal integer k such that $\tau^{k/2}r_1 \leq r_2$. By definition, put $\rho_i = \tau^{-i/2}r_2$, $i = 0, \dots, k-1$, and $\rho_k = r_1$. We obviously have

$$\tau^{1/2}\rho_{i+1} \leq \rho_i \leq \tau\rho_{i+1}, \quad i = 0, \dots, k-1.$$

As in the proof of Lemma 3.4, let Ξ_1 be the set of integers $i \in \{0, \dots, k-1\}$ such that $M(\rho_i; u) > M(\rho_{i+1} + 0; u)$ and, moreover, condition (3.5) is fulfilled. Also denote $\Xi_2 = \{0, \dots, k-1\} \setminus \Xi_1$.

From Lemma 3.2, it follows that

$$\begin{aligned} &(M(\rho_i; u) - M(\rho_{i+1} + 0; u)) \sup_{(\eta^{-1/2}M(\rho_{i+1}+0;u), M(\rho_i;u))} g^{-1/(p-1)} \\ &\geq c(\rho_i - \rho_{i+1})^{p/(p-1)} \operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q^{1/(p-1)} \end{aligned}$$

for all $i \in \Xi_1$. Combining this with the evident inequalities

$$\begin{aligned} \int_{M(\rho_{i+1}+0;u)}^{M(\rho_i;u)} g_\eta^{-1/(p-1)}(t) dt &\geq (M(\rho_i; u) - M(\rho_{i+1} + 0; u)) \\ &\quad \times \sup_{(\eta^{-1/2}M(\rho_{i+1}+0;u), M(\rho_i;u))} g^{-1/(p-1)} \end{aligned}$$

and

$$(\rho_i - \rho_{i+1})^{p/(p-1)} \operatorname{ess\,inf}_{\Omega_{\rho_{i+1}, \rho_i}} q^{1/(p-1)} \geq c \int_{\rho_{i+1}}^{\rho_i} (\xi f_\tau(\xi))^{1/(p-1)} d\xi,$$

we obtain

$$\int_{M(\rho_{i+1}+0;u)}^{M(\rho_i;u)} g_{\eta}^{-1/(p-1)}(t) dt \geq c \int_{\rho_{i+1}}^{\rho_i} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi.$$

Summing the last inequality over all $i \in \Xi_1$, one can conclude that

$$\int_{M(\rho_k+0;u)}^{M(\rho_0;u)} g_{\eta}^{-1/(p-1)}(t) dt \geq c \sum_{i \in \Xi_1} \int_{\rho_{i+1}}^{\rho_i} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi. \quad (3.16)$$

If

$$\sum_{i \in \Xi_1} \int_{\rho_{i+1}}^{\rho_i} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi \geq \frac{1}{2} \int_{r_1}^{r_2} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi \quad (3.17)$$

then (3.16) immediately implies (3.15). Assume that (3.17) is not valid, then

$$\sum_{i \in \Xi_2} \int_{\rho_{i+1}}^{\rho_i} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi \geq \frac{1}{2} \int_{r_1}^{r_2} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi. \quad (3.18)$$

Repeating the arguments given in the proof of inequality (3.13) with r_1 and r_2 replaced by ρ_{i+1} and ρ_i , respectively, we have

$$\begin{aligned} \int_{M(\rho_{i+1}+0;u)}^{M(\rho_i;u)} g_{\eta}^{-1/\alpha}(t) dt &\geq c(\rho_i - \rho_{i+1}) \sup_{r \in (\rho_{i+1}, \rho_i)} (r f_{\tau}(r))^{1/(p-1)} \\ &\geq c \int_{\rho_{i+1}}^{\rho_i} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi \end{aligned}$$

for all $i \in \Xi_2$, whence it follows that

$$\int_{M(\rho_k+0;u)}^{M(\rho_0;u)} g_{\eta}^{-1/\alpha}(t) dt \geq c \sum_{i \in \Xi_2} \int_{\rho_{i+1}}^{\rho_i} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi.$$

Combining this with (3.18) we obtain estimate (3.2).

The proof is completed. \square

Lemma 3.6. *Let $M_1 \leq M(r_1 + 0; u) \leq M(r_2; u) \leq M_2$, $\sigma r_1 \leq r_2$, and $\theta M_1 \leq M_2$, where $r_0 \leq r_1 < r_2$, $0 < M_1 < M_2$, $\sigma > 1$, and $\theta > 1$ are some real numbers. Then*

$$\begin{aligned} \left(\int_{M_1}^{M_2} (g_{\theta}(t)t)^{-1/p} dt \right)^{p/(p-1)} + \int_{M_1}^{M_2} g_{\theta}^{-1/\alpha}(t) dt \\ \geq C \int_{r_1}^{r_2} (\xi f_{\sigma}(\xi))^{1/(p-1)} d\xi, \end{aligned} \quad (3.19)$$

where the constant $C > 0$ depends only on n , p , α , θ , σ , C_1 , and C_2 .

Proof. We denote $\eta = \theta^{1/2}$ and $\tau = \sigma^{1/2}$. Let k be the maximal integer satisfying the condition $\tau^{k/2} r_1 \leq r_2$. We put $\xi_i = \tau^{i/2} r_1$, $i = 0, \dots, k-1$, and $\xi_k = r_2$. It is obvious that

$$\tau^{1/2} \xi_i \leq \xi_{i+1} \leq \tau \xi_i, \quad i = 0, \dots, k-1. \quad (3.20)$$

In addition, for any $i \in \{0, \dots, k-1\}$ at least one of the following three inequalities holds:

$$\int_{M(\xi_i+0;u)}^{M(\xi_{i+1};u)} (g_{\eta}(t)t)^{-1/p} dt \geq c \int_{\xi_i}^{\xi_{i+1}} q_{\tau}^{1/p}(\xi) d\xi, \quad (3.21)$$

$$\int_{M(\xi_i+0;u)}^{M(\xi_{i+1};u)} g_{\eta}^{-1/(p-1)}(t) dt \geq c \int_{\xi_i}^{\xi_{i+1}} (\xi f_{\tau}(\xi))^{1/(p-1)} d\xi, \quad (3.22)$$

$$\int_{M(\xi_i+0;u)}^{M(\xi_{i+1};u)} g_\eta^{-1/\alpha}(t) dt \geq c \int_{\xi_i}^{\xi_{i+1}} (\xi f_\tau(\xi))^{1/(p-1)} d\xi. \quad (3.23)$$

This follows from Lemma 3.4 if $\eta^{1/2}M(\xi_i+0;u) \leq M(\xi_{i+1};u)$ or from Lemma 3.5 otherwise.

By Ξ_1 , Ξ_2 , and Ξ_3 we denote the sets of integers $i \in \{0, \dots, k-1\}$ satisfying relations (3.21), (3.22), and (3.23), respectively. Let

$$\sum_{i \in \Xi_3} \int_{\xi_i}^{\xi_{i+1}} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \geq \frac{1}{3} \int_{r_1}^{r_2} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi. \quad (3.24)$$

Summing (3.23) over all $i \in \Xi_3$, we obtain

$$\int_{M(r_1+0;u)}^{M(r_2;u)} g_\eta^{-1/\alpha}(t) dt \geq c \sum_{i \in \Xi_3} \int_{\xi_i}^{\xi_{i+1}} (\xi f_\tau(\xi))^{1/(p-1)} d\xi.$$

The last inequality and (3.24) obviously imply (3.19) since $g_\tau(t) \geq g_\theta(t)$ for all $t > 0$ and, moreover, $f_\tau(\xi) \geq f_\sigma(\xi)$ for all $\xi \in (r_1, r_2)$. Assume that (3.24) does not hold. In this case, we have either

$$\sum_{i \in \Xi_1} \int_{\xi_i}^{\xi_{i+1}} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \geq \frac{1}{3} \int_{r_1}^{r_2} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \quad (3.25)$$

or

$$\sum_{i \in \Xi_2} \int_{\xi_i}^{\xi_{i+1}} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \geq \frac{1}{3} \int_{r_1}^{r_2} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi. \quad (3.26)$$

At first, let (3.25) be valid. Taking into account (3.20), we obtain

$$\begin{aligned} \left(\sum_{i \in \Xi_1} \int_{\xi_i}^{\xi_{i+1}} q_\tau^{1/p}(\xi) d\xi \right)^{p/(p-1)} &\geq \sum_{i \in \Xi_1} \left(\int_{\xi_i}^{\xi_{i+1}} q_\tau^{1/p}(\xi) d\xi \right)^{p/(p-1)} \\ &\geq \sum_{i \in \Xi_1} (\xi_{i+1} - \xi_i)^{p/(p-1)} \inf_{(\xi_{i+1}, \xi_i)} q_\tau^{1/(p-1)} \\ &\geq c \sum_{i \in \Xi_1} \int_{\xi_i}^{\xi_{i+1}} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi. \end{aligned}$$

By (3.21) and (3.25), this yields the estimate

$$\left(\int_{M(r_1+0;u)}^{M(r_2;u)} (g_\eta(t)t)^{-1/p} dt \right)^{p/(p-1)} \geq c \int_{r_1}^{r_2} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi,$$

whence (3.19) follows at once.

Now, let (3.26) hold. Then, summing (3.22) over all $i \in \Xi_2$, we conclude that

$$\int_{M(r_1+0;u)}^{M(r_2;u)} g_\eta^{-1/(p-1)}(t) dt \geq c \int_{r_1}^{r_2} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi. \quad (3.27)$$

Take the maximal integer l satisfying the condition $\eta^{l/2}M_1 \leq M_2$. We denote $t_i = \eta^{i/2}M_1$, $i = 0, \dots, l-1$, and $t_l = M_2$. It can easily be seen that

$$\eta^{1/2}t_i \leq t_{i+1} \leq \eta t_i, \quad i = 0, \dots, l-1.$$

We have

$$\begin{aligned}
\left(\int_{M_1}^{M_2} (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} &= \left(\sum_{i=0}^{l-1} \int_{t_i}^{t_{i+1}} (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} \\
&\geq \sum_{i=0}^{l-1} \left(\int_{t_i}^{t_{i+1}} (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} \\
&\geq \sum_{i=0}^{l-1} (t_{i+1} - t_i)^{p/(p-1)} t_{i+1}^{-1/(p-1)} \inf_{(t_i, t_{i+1})} g_\theta^{-1/(p-1)},
\end{aligned}$$

whence in accordance with the inequality

$$(t_{i+1} - t_i)^{p/(p-1)} t_{i+1}^{-1/(p-1)} \inf_{(t_i, t_{i+1})} g_\theta^{-1/(p-1)} \geq c \int_{t_i}^{t_{i+1}} g_\eta^{-1/(p-1)}(t) dt$$

it follows that

$$\begin{aligned}
\left(\int_{M_1}^{M_2} (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} &\geq c \sum_{i=0}^{l-1} \int_{t_i}^{t_{i+1}} g_\eta^{-1/(p-1)}(t) dt \\
&= c \int_{M_1}^{M_2} g_\eta^{-1/(p-1)}(t) dt.
\end{aligned}$$

By (3.27), this implies the estimate

$$\left(\int_{M_1}^{M_2} (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} \geq c \int_{r_1}^{r_2} (\xi f_\sigma(\xi))^{1/(p-1)} d\xi$$

from which we immediately obtain (3.19).

Lemma 3.6 is completely proved. \square

Proof of Theorem 2.1. Assume to the contrary, that u is a non-negative solution of (1.1), (1.2) and, moreover, $M(r_*; u) > 0$ for some $r_* > r_0$. Lemma 3.6 and condition (2.3) allows us to assert that $M(r; u) \rightarrow \infty$ as $r \rightarrow \infty$. In so doing, the inequality

$$\begin{aligned}
\left(\int_{M(r_*; u)}^{M(r; u)} (g_\theta(t)t)^{-1/p} dt \right)^{p/(p-1)} &+ \int_{M(r_*; u)}^{M(r; u)} g_\theta^{-1/\alpha}(t) dt \\
&\geq C \int_{r_*}^r (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \quad (3.28)
\end{aligned}$$

holds for all sufficiently large r , where the constant $C > 0$ depends only on $n, p, \alpha, \theta, \sigma, C_1$, and C_2 . Passing in this inequality to the limit as $r \rightarrow \infty$, we get a contradiction to conditions (2.1)–(2.3).

Theorem 2.1 is completely proved. \square

Proof of Theorem 2.2. As in the proof of Theorem 2.1, we have $M(r; u) \rightarrow \infty$ as $r \rightarrow \infty$. Hence, in formula (3.28), the real number r_* can be taken such that $M(r_*; u) > 1$. According to (2.3), we also have

$$\int_{r_*}^r (\xi f_\sigma(\xi))^{1/(p-1)} d\xi \geq \frac{1}{2} \int_{r_0}^r (\xi f_\sigma(\xi))^{1/(p-1)} d\xi$$

for all sufficiently large r . Thus, estimate (2.21) follows at once from (3.28).

Theorem 2.2 is completely proved. \square

Proof of Theorem 2.3. If $u \equiv 0$, then (2.22) is evident. Let $u \not\equiv 0$. In this case, it is obvious that $M(r; u) > 0$ for all r in a neighborhood of infinity since $M(\cdot; u)$ is a non-decreasing function. Consequently, applying Lemma 3.6, we obtain

$$\begin{aligned} \left(\int_{M(r; u)}^{\infty} (g_{\theta}(t)t)^{-1/p} dt \right)^{p/(p-1)} + \int_{M(r; u)}^{\infty} g_{\theta}^{-1/\alpha}(t) dt \\ \geq C \int_r^{\infty} (\xi f_{\sigma}(\xi))^{1/(p-1)} d\xi \end{aligned}$$

for all sufficiently large r , where the constant $C > 0$ depends only on $n, p, \alpha, \theta, \sigma, C_1$, and C_2 .

To complete the proof, it remains to note that the last inequality is equivalent to (2.22). \square

4. PROOF OF LEMMAS 3.1–3.3

We extend the functions A and b in the left-hand side of (1.1) by putting $A(x, \xi) = (C_1 + C_2)|\xi|^{p-2}\xi/2$ and $b(x) = 0$ for all $x \in \mathbb{R}^n \setminus \Omega, \xi \in \mathbb{R}^n$.

Let us agree on the following notation. In the proof of Lemma 4.2, by c we denote various positive constants that can depend only on n, p, α , and ω . In the proof of Lemmas 4.4, 4.5, 4.6, and 4.8, analogous constants can depend only on n, p, α, C_1 , and C_2 , whereas in the proof of Lemmas 4.7 and 4.9, they can depend only on $n, p, \alpha, \gamma, C_1$, and C_2 . Finally, in the proof of Lemma 4.11 these constants can depend only on $n, p, \alpha, \gamma, \lambda, C_1$, and C_2 .

Assume that ω_1 and ω_2 are open subsets of \mathbb{R}^n . A function $v \in W_{p,loc}^1(\omega_1 \cap \omega_2)$ satisfies the condition

$$v|_{\omega_2 \cap \partial\omega_1} = 0$$

if $\psi v \in \dot{W}_p^1(\omega_1 \cap \omega_2)$ for any $\psi \in C_0^\infty(\omega_2)$. We also say that

$$v|_{\omega_2 \cap \partial\omega_1} \leq 0 \tag{4.1}$$

if $\psi \max\{v, 0\} \in \dot{W}_p^1(\omega_1 \cap \omega_2)$ for any $\psi \in C_0^\infty(\omega_2)$.

Lemma 4.1. *Let $v \in W_{p,loc}^1(\omega_1 \cap \omega_2)$ be a solution of the inequality*

$$\operatorname{div} A(x, Dv) + b(x)|Dv|^\alpha \geq a(x) \quad \text{in } \omega_1 \cap \omega_2 \tag{4.2}$$

satisfying condition (4.1), where $a \in L_{1,loc}(\omega_1 \cap \omega_2)$. We denote $\tilde{\omega} = \{x \in \omega_1 \cap \omega_2 : v(x) > 0\}$,

$$\tilde{v}(x) = \begin{cases} v(x), & x \in \tilde{\omega}, \\ 0, & x \in \omega_2 \setminus \tilde{\omega} \end{cases}$$

and

$$\tilde{a}(x) = \begin{cases} a(x), & x \in \tilde{\omega}, \\ 0, & x \in \omega_2 \setminus \tilde{\omega}. \end{cases}$$

Then

$$\operatorname{div} A(x, D\tilde{v}) + b(x)|D\tilde{v}|^\alpha \geq \tilde{a}(x) \quad \text{in } \omega_2.$$

Proof. We take a non-decreasing function $\eta \in C^\infty(\mathbb{R})$ such that $\eta|_{(-\infty, 0]} = 0$ and $\eta|_{[1, \infty)} = 1$. Put $\eta_\tau(t) = \eta(t/\tau)$ and $\varphi = \psi\eta_\tau(v)$, where $\psi \in C_0^\infty(\omega_2)$ is a non-negative function and $\tau > 0$ is a real number. By (4.2), we have

$$-\int_{\omega_1 \cap \omega_2} A(x, Dv) D\varphi dx + \int_{\omega_1 \cap \omega_2} b(x)|Dv|^\alpha \varphi dx \geq \int_{\omega_1 \cap \omega_2} a(x)\varphi dx,$$

whence it follows that

$$\begin{aligned}
& - \int_{\omega_1 \cap \omega_2} \eta_\tau(u) A(x, Dv) D\psi \, dx + \int_{\omega_1 \cap \omega_2} b(x) |Dv|^\alpha \eta_\tau(u) \psi \, dx \\
& \geq \int_{\omega_1 \cap \omega_2} a(x) \psi \eta_\tau(u) \, dx + \int_{\omega_1 \cap \omega_2} \eta'_\tau(u) \psi A(x, Dv) Dv \, dx \\
& \geq \int_{\omega_1 \cap \omega_2} a(x) \psi \eta_\tau(u) \, dx.
\end{aligned}$$

Passing to the limit as $\tau \rightarrow +0$ in the last expression, we obtain

$$\begin{aligned}
& - \int_{\omega_1 \cap \omega_2} \chi_{\tilde{\omega}}(x) A(x, Dv) D\psi \, dx + \int_{\omega_1 \cap \omega_2} b(x) |Dv|^\alpha \chi_{\tilde{\omega}}(x) \psi \, dx \\
& \geq \int_{\omega_1 \cap \omega_2} \chi_{\tilde{\omega}}(x) a(x) \psi \, dx,
\end{aligned}$$

where $\chi_{\tilde{\omega}}$ is the characteristic function of the set $\tilde{\omega}$, i.e. $\chi_{\tilde{\omega}}(x) = 1$ if $x \in \tilde{\omega}$ and $\chi_{\tilde{\omega}}(x) = 0$ otherwise. Since

$$D\tilde{v}(x) = \begin{cases} Dv(x), & x \in \tilde{\omega}, \\ 0, & x \in \omega_2 \setminus \tilde{\omega}, \end{cases}$$

this immediately implies that

$$- \int_{\omega_2} A(x, D\tilde{v}) D\psi \, dx + \int_{\omega_2} b(x) |D\tilde{v}|^\alpha \psi \, dx \geq \int_{\omega_2} \tilde{a}(x) \psi \, dx.$$

The proof is completed. \square

Corollary 4.1. *Let $v \in W_p^1(\omega)$ be a solution of the inequality*

$$\operatorname{div} A(x, Dv) + b(x) |Dv|^\alpha \geq a(x) \quad \text{in } \omega,$$

where ω is an open subset of \mathbb{R}^n and $a \in L_{1,loc}(\omega)$. If $\tilde{\omega} = \{x \in \omega : v(x) > 0\}$, $\tilde{v} = \chi_{\tilde{\omega}} v$ and, moreover, $\tilde{a} = \chi_{\tilde{\omega}} a$, then

$$\operatorname{div} A(x, D\tilde{v}) + b(x) |D\tilde{v}|^\alpha \geq \tilde{a}(x) \quad \text{in } \omega.$$

Proof. We put $\omega_1 = \omega_2 = \omega$ in Lemma 4.1. \square

Lemma 4.2 (the maximum principle). *Let $v \in W_p^1(\omega) \cap L_\infty(\omega)$ be a non-negative solution of the inequality*

$$\operatorname{div} A(x, Dv) + b(x) |Dv|^\alpha \geq 0 \quad \text{in } \omega, \tag{4.3}$$

where $\omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary. Then

$$\operatorname{ess\,sup}_\omega v = \operatorname{ess\,sup}_{\partial\omega} v, \tag{4.4}$$

where the restriction of v to $\partial\omega$ is understood in the sense of the trace and the $\operatorname{ess\,sup}$ on the right-hand side is with respect to $(n-1)$ -dimensional Lebesgue measure on $\partial\omega$.

Proof. Assume that (4.4) is not valid. We put

$$v_\tau(x) = \max\{v(x) - \tau, 0\},$$

where τ is a real number satisfying the condition

$$\operatorname{ess\,sup}_{\partial\omega} v < \tau < \operatorname{ess\,sup}_\omega v. \tag{4.5}$$

It can easily be seen that v_τ is a non-negative function belonging to $W_p^1(\omega)$; therefore,

$$-\int_{\omega} A(x, Dv) Dv_\tau dx + \int_{\omega} b(x) |Dv|^\alpha v_\tau dx \geq 0 \quad (4.6)$$

in accordance with (4.3). Since

$$Dv_\tau(x) = \begin{cases} Dv(x), & x \in \omega_\tau, \\ 0, & x \in \omega \setminus \omega_\tau, \end{cases}$$

where $\omega_\tau = \{x \in \omega : \tau < v(x) < \text{ess sup}_\omega v\}$, we have

$$\int_{\omega} A(x, Dv) Dv_\tau dx = \int_{\omega_\tau} A(x, Dv_\tau) Dv_\tau dx \geq C_1 \int_{\omega_\tau} |Dv_\tau|^p dx$$

and

$$\int_{\omega} b(x) |Dv|^\alpha v_\tau dx = \int_{\omega_\tau} b(x) |Dv_\tau|^\alpha v_\tau dx \leq \text{ess sup}_\omega b \int_{\omega_\tau} |Dv_\tau|^\alpha v_\tau dx.$$

Hence, (4.6) allows us to assert that

$$C_1 \int_{\omega_\tau} |Dv_\tau|^p dx \leq \text{ess sup}_\omega b \int_{\omega_\tau} |Dv_\tau|^\alpha v_\tau dx \quad (4.7)$$

At first, consider the case of $p-1 \leq \alpha < p$. Let $p_1 = p/\alpha$ and p_2 be some real number such that $p < (p-\alpha)p_2 < np/(n-p)$ if $n > p$ and $p < (p-\alpha)p_2$ if $n \leq p$. We also take the real number p_3 satisfying the relation $1/p_1 + 1/p_2 + 1/p_3 = 1$. Since $1/p_1 + 1/p_2 < 1$, we obviously have $p_3 > 1$.

From the Hölder inequality for three functions, it follows that

$$\begin{aligned} \int_{\omega_\tau} |Dv_\tau|^\alpha v_\tau dx &\leq \left(\int_{\omega_\tau} |Dv_\tau|^p dx \right)^{1/p_1} \left(\int_{\omega_\tau} v_\tau^{(p-\alpha)p_2} dx \right)^{1/p_2} \\ &\quad \times \left(\int_{\omega_\tau} v_\tau^{(1-p+\alpha)p_3} dx \right)^{1/p_3}. \end{aligned} \quad (4.8)$$

Using the Friedrichs inequality and the Sobolev embedding theorem [15], we obtain

$$\left(\int_{\omega} v_\tau^{(p-\alpha)p_2} dx \right)^{1/p_2} \leq c \left(\int_{\omega} |Dv_\tau|^p dx \right)^{(p-\alpha)/p}$$

It is also obvious that

$$\int_{\omega} v_\tau^{(p-\alpha)p_2} dx = \int_{\omega_\tau} v_\tau^{(p-\alpha)p_2} dx$$

and

$$\int_{\omega} |Dv_\tau|^p dx = \int_{\omega_\tau} |Dv_\tau|^p dx.$$

Consequently, (4.8) implies the estimate

$$\int_{\omega_\tau} |Dv_\tau|^\alpha v_\tau dx \leq c \left(\int_{\omega_\tau} v_\tau^{(1-p+\alpha)p_3} dx \right)^{1/p_3} \int_{\omega_\tau} |Dv_\tau|^p dx.$$

Since $\text{mes } \omega_\tau \rightarrow 0$ as $\tau \rightarrow \text{ess sup}_\omega v - 0$, we have

$$\int_{\omega_\tau} v_\tau^{(1-p+\alpha)p_3} dx \leq \text{mes } \omega_\tau \text{ess sup}_{\omega_\tau} v_\tau^{(1-p+\alpha)p_3} \rightarrow 0 \quad \text{as } \tau \rightarrow \text{ess sup}_\omega v - 0.$$

Thus, by appropriate choice of the real number τ , one can achieve that

$$\operatorname{ess\,sup}_{\omega} b \int_{\omega_{\tau}} |Dv_{\tau}|^{\alpha} v_{\tau} dx \leq \frac{C_1}{2} \int_{\omega_{\tau}} |Dv_{\tau}|^p dx. \quad (4.9)$$

Combining the last estimate with (4.7), we obtain

$$\int_{\omega_{\tau}} |Dv_{\tau}|^p dx \leq 0, \quad (4.10)$$

whence it follows that $v_{\tau} = 0$ since $v_{\tau} \in \mathring{W}_p^1(\omega)$. This contradicts (4.5).

Now, let $\alpha = p$. Then

$$\int_{\omega_{\tau}} |Dv_{\tau}|^{\alpha} v_{\tau} dx \leq \operatorname{ess\,sup}_{\omega_{\tau}} v_{\tau} \int_{\omega_{\tau}} |Dv_{\tau}|^p dx.$$

It is easy to see that

$$\operatorname{ess\,sup}_{\omega_{\tau}} v_{\tau} = \operatorname{ess\,sup}_{\omega} v - \tau \rightarrow 0 \quad \text{as } \tau \rightarrow \operatorname{ess\,sup}_{\omega} v - 0;$$

therefore, taking the real number τ close enough to $\operatorname{ess\,sup}_{\omega} v$, we again obtain inequality (4.9) which immediately implies (4.10). Thus, we derive a contradiction once more.

The proof is completed. \square

Corollary 4.2. *Let $u \geq 0$ be a solution of problem (1.1), (1.2), then*

$$M(r; u) = \operatorname{ess\,sup}_{B_r \cap \Omega} u \quad (4.11)$$

for all $r > r_0$.

Proof. We extend the function u on the whole set \mathbb{R}^n by putting $u = 0$ on $\mathbb{R}^n \setminus \Omega$. According to Lemma 4.1, the extended function \tilde{u} satisfies the inequality

$$\operatorname{div} A(x, D\tilde{u}) + b(x)|D\tilde{u}|^{\alpha} \geq 0 \quad \text{in } \mathbb{R}^n.$$

Thus, to obtain (4.11), it remains to use Lemma 4.2 with $\omega = B_r$. \square

By Q_l^z we mean the open cube in \mathbb{R}^n of edge length $l > 0$ and center at a point z . In the case of $z = 0$, we write Q_l instead of Q_l^0 .

The following assertion is elementary, but useful.

Lemma 4.3. *Let $v \in W_p^1(Q_l^z)$, $l > 0$, $z \in \mathbb{R}^n$. If $\operatorname{mes}\{x \in Q_l^z : v(x) = 0\} \geq l^n/2$, then*

$$\int_{Q_l^z} v^p dx \leq Cl^p \int_{Q_l^z} |Dv|^p dx,$$

where the constant $C > 0$ depends only on n and p .

Proof. Without loss of generality, it can be assumed that $l = 1$ and $z = 0$; otherwise we use the change of coordinates.

The proof is by reductio ad absurdum. Let there be a sequence $v_k \in W_p^1(Q_1)$ such that $\operatorname{mes}\{x \in Q_1 : v_k(x) = 0\} \geq 1/2$ and

$$\int_{Q_1} v_k^p dx > k \int_{Q_1} |Dv_k|^p dx, \quad k = 1, 2, \dots \quad (4.12)$$

It can be assumed that

$$\int_{Q_1} v_k^p dx = 1, \quad k = 1, 2, \dots; \quad (4.13)$$

otherwise we replace v_k by $v_k/\|v_k\|_{L_p(Q_1)}$. Therefore, $\{v_k\}_{k=0}^\infty$ is a bounded sequence in $W_p^1(Q_1)$. By the Sobolev embedding theorem [15], it has a subsequence that converges in $L_p(Q_1)$. Taking into account (4.12) and (4.13), one can obviously claim that this subsequence converges in the space $W_p^1(Q_1)$ to a real number $\lambda \neq 0$. To reduce clutter in indices, we also denote this subsequence by $\{v_k\}_{k=0}^\infty$. Thus, we have

$$\|v_k - \lambda\|_{L_p(Q_1)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

At the same time, from the definition of the functions v_k , it follows that

$$\|v_k - \lambda\|_{L_p(Q_1)} \geq |\lambda| \operatorname{mes}\{x \in Q_1 : v_k(x) = 0\} \geq \frac{|\lambda|}{2}, \quad k = 1, 2, \dots$$

This contradiction proves the lemma. \square

Lemma 4.4. *Let $v \in W_p^1(B_r^z) \cap L_\infty(B_r^z)$, $r > 0$, $z \in \mathbb{R}^n$, be a non-negative solution of the inequality*

$$\operatorname{div} A(x, Dv) + b(x)|Dv|^\alpha \geq 0 \quad \text{in } B_r^z$$

satisfying the condition

$$r^{p-\alpha} \left(\operatorname{ess\,sup}_{B_r^z} v \right)^{\alpha-p+1} \operatorname{ess\,sup}_{B_r^z} b \leq \frac{C_1}{2}. \quad (4.14)$$

Then

$$\int_{B_{r_1}^z} |Dv|^\gamma dx \leq \frac{C}{(r_2 - r_1)^p} \int_{B_{r_2}^z} v^\gamma dx \quad (4.15)$$

for all real numbers $0 < r_1 < r_2 \leq r$ and $\gamma \geq 1$, where the constant $C > 0$ depends only on n, p, α, C_1 , and C_2 .

Proof. We denote

$$m = \operatorname{ess\,sup}_{B_r^z} v, \quad \mu = \operatorname{ess\,sup}_{B_r^z} b, \quad (4.16)$$

and $\psi(x) = \psi_0(|x - z| - r_1)/(r_2 - r_1)$, where $\psi_0 \in C^\infty(\mathbb{R})$ is a non-increasing function such that $\psi_0|_{(-\infty, 0]} = 1$ and $\psi_0|_{[1, \infty)} = 0$. It can be assumed that $m > 0$; otherwise (4.15) is evident. We also put $\beta = \gamma - (p - 1)/p$. It is easy to see that $\beta \geq 1/p$. Taking $\varphi = \psi^p v^{\beta p}$ in the integral inequality

$$-\int_{B_r^z} A(x, Dv) D\varphi dx + \int_{B_r^z} b(x)|Dv|^\alpha \varphi dx \geq 0,$$

we obtain

$$\begin{aligned} p\beta \int_{B_r^z} \psi^p v^{\beta p-1} A(x, Dv) Dv dx &\leq -p \int_{B_r^z} \psi^{p-1} v^{\beta p} A(x, Dv) D\psi dx \\ &\quad + \int_{B_r^z} b(x)|Dv|^\alpha \psi^p v^{\beta p} dx. \end{aligned}$$

Since

$$C_1 \int_{B_r^z} \psi^p v^{\beta p-1} |Dv|^p dx \leq \int_{B_r^z} \psi^p v^{\beta p-1} A(x, Dv) Dv dx$$

and

$$\int_{B_r^z} \psi^{p-1} v^{\beta p} |A(x, Dv)| |D\psi| dx \leq C_2 \int_{B_r^z} \psi^{p-1} v^{\beta p} |Dv|^{p-1} |D\psi| dx,$$

this implies the estimate

$$\begin{aligned} p\beta C_1 \int_{B_r^z} \psi^p v^{\beta p-1} |Dv|^p dx &\leq pC_2 \int_{B_r^z} \psi^{p-1} v^{\beta p} |Dv|^{p-1} |D\psi| dx \\ &\quad + \int_{B_r^z} b(x) |Dv|^\alpha \psi^p v^{\beta p} dx \end{aligned}$$

from which, denoting $w(y) = (v(ry + z)/m)^\gamma$ and $\eta(y) = \psi(ry + z)$, we have

$$\begin{aligned} \frac{p\beta C_1}{\gamma^p} \int_{B_1} \eta^p |Dw|^p dy &\leq \frac{pC_2}{\gamma^{p-1}} \int_{B_1} \eta^{p-1} w |Dw|^{p-1} |D\eta| dy \\ &\quad + \frac{r^{p-\alpha} m^{\alpha-p+1} \mu}{\gamma^\alpha} \int_{B_1} |Dw|^\alpha \eta^p w^{(\alpha-p+1)/\gamma+p-\alpha} dy \\ &\leq \frac{pC_2}{\gamma^{p-1}} \int_{B_1} \eta^{p-1} w |Dw|^{p-1} |D\eta| dy \\ &\quad + \frac{C_1}{2\gamma^\alpha} \int_{B_1} |Dw|^\alpha \eta^p w^{(\alpha-p+1)/\gamma+p-\alpha} dy \end{aligned} \quad (4.17)$$

in accordance with condition (4.14). From the Young inequality, it follows that

$$\begin{aligned} \frac{pC_2}{\gamma^{p-1}} \int_{B_1} \eta^{p-1} w |Dw|^{p-1} |D\eta| dy &\leq \frac{p\beta C_1}{4\gamma^p} \int_{B_1} \eta^p |Dw|^p dy \\ &\quad + \frac{c}{\beta^{p-1}} \int_{B_1} w^p |D\eta|^p dy. \end{aligned}$$

In the case of $\alpha < p$, the Young inequality also implies the relation

$$\begin{aligned} \frac{C_1}{2\gamma^\alpha} \int_{B_1} |Dw|^\alpha \eta^p w^{(\alpha-p+1)/\gamma+p-\alpha} dy &\leq \frac{p\beta C_1}{2\gamma^p} \int_{B_1} \eta^p |Dw|^p dy \\ &\quad + \frac{c}{\beta^{\alpha/(p-\alpha)}} \int_{B_1} \eta^p w^{(\alpha-p+1)p/(\gamma(p-\alpha))+p} dy \\ &\leq \frac{p\beta C_1}{2\gamma^p} \int_{B_1} \eta^p |Dw|^p dy \\ &\quad + \frac{c}{\beta^{\alpha/(p-\alpha)}} \int_{B_1} \eta^p w^p dy. \end{aligned}$$

In turn, for $\alpha = p$, one can assert that

$$\begin{aligned} \frac{C_1}{2\gamma^\alpha} \int_{B_1} |Dw|^\alpha \eta^p w^{(\alpha-p+1)/\gamma+p-\alpha} dy &= \frac{C_1}{2\gamma^\alpha} \int_{B_1} \eta^p |Dw|^p w^{1/\gamma} dx \\ &\leq \frac{C_1}{2\gamma^p} \int_{B_1} \eta^p |Dw|^p dx. \end{aligned}$$

Therefore, taking into account (4.17) and the fact that $\beta = \gamma - (p-1)/p \geq 1/p$, we obtain

$$\int_{B_r} \eta^p |Dw|^p dy \leq c \int_{B_r} (\eta^p + |D\eta|^p) w^p dy.$$

This implies the estimate

$$\int_{B_{r_1/r}} |Dw|^p dy \leq c \|\psi_0\|_{C^1(\mathbb{R})} \left(\frac{r}{r_2 - r_1} \right)^p \int_{B_{r_2/r}} w^p dy,$$

whence (4.15) follows at once.

The proof is completed. \square

Lemma 4.5 (Moser's inequality). *Under the hypotheses of Lemma 4.4, we have*

$$\operatorname{ess\,sup}_{B_{r/2}^z} v \leq Cr^{-n/p} \left(\int_{B_r^z} v^p dx \right)^{1/p}, \quad (4.18)$$

where the constant $C > 0$ depends only on n, p, α, C_1 , and C_2 .

Proof. We take a real number λ satisfying the conditions $1 < \lambda < n/(n-p)$ in the case of $n > p$ and $1 < \lambda$ in the case of $n \leq p$. Also denote $r_k = 1/2 + 1/2^{k+1}$ and $p_k = \lambda^k p$, $k = 0, 1, 2, \dots$

We use Moser's iterative process. By Lemma 4.4,

$$\|Dv^{\lambda^k}\|_{L_p(B_{r_{k+1}}^z)} \leq \frac{2^k c}{r} \|v^{\lambda^k}\|_{L_p(B_{r_k}^z)}$$

for any non-negative integer k , whence it follows that

$$\|Dw^{\lambda^k}\|_{L_p(B_{r_{k+1}})} \leq 2^k c \|w^{\lambda^k}\|_{L_p(B_{r_k})},$$

where $w(y) = v(ry + z)$. This implies the inequality

$$\|w^{\lambda^k}\|_{W_p^1(B_{r_{k+1}})} \leq 2^k c \|w^{\lambda^k}\|_{L_p(B_{r_k})}.$$

At the same time,

$$\|w^{\lambda^k}\|_{L_{\lambda p}(B_{r_{k+1}})} \leq c \|w^{\lambda^k}\|_{W_p^1(B_{r_{k+1}})}$$

by the Sobolev embedding theorem [15]. Thus, we obtain

$$\|w^{\lambda^k}\|_{L_{\lambda p}(B_{r_{k+1}})} \leq 2^k c \|w^{\lambda^k}\|_{L_p(B_{r_k})}$$

or, in other words,

$$\|w\|_{L_{p_{k+1}}(B_{r_{k+1}})} \leq (2^k c)^{\lambda^{-k}} \|w\|_{L_{p_k}(B_{r_k})}, \quad k = 0, 1, 2, \dots$$

The last formula allows us to assert that

$$\|w\|_{L_\infty(B_{1/2})} \leq c \|w\|_{L_p(B_1)},$$

whence we immediately derive (4.18).

The proof is completed. \square

Lemma 4.6. *Under the hypotheses of Lemma 4.4, for any real number $\varepsilon > 0$ there is a real number $\delta > 0$ depending only on $n, p, \alpha, \varepsilon, C_1$, and C_2 such that the condition $\operatorname{mes}\{x \in B_r^z : v(x) > 0\} \leq \delta r^n$ implies the inequality*

$$\operatorname{ess\,sup}_{B_{r/8}^z} v \leq \varepsilon \operatorname{ess\,sup}_{B_r^z} v. \quad (4.19)$$

Proof. We denote $\omega = \operatorname{mes}\{x \in B_r^z : v(x) > 0\}$. In the case of $\operatorname{mes}\omega = 0$, inequality (4.19) is obvious; therefore, it can be assumed that $\operatorname{mes}\omega > 0$.

By Lemma 4.4,

$$\int_{B_{r/2}^z} |Dv|^p dx \leq cr^{-p} \int_{B_r^z} v^p dx. \quad (4.20)$$

Let $\text{mes } \omega < r^n/2^{n+1}$ and, moreover, k be the maximal positive integer such that $\text{mes } \omega < r^n/(2^{n+1}k^n)$. We take a family of disjoint open cubes $Q_{r/(2k)}^{z_i}$, $i = 1, \dots, k^n$, satisfying the condition $\overline{Q_{r/2}^z} = \bigcup_{i=1}^{k^n} \overline{Q_{r/(2k)}^{z_i}}$. From Lemma 4.3, it follows that

$$\int_{Q_{r/(2k)}^{z_i}} v^p dx \leq ck^{-p}r^p \int_{Q_{r/(2k)}^{z_i}} |Dv|^p dx, \quad i = 1, \dots, k^n;$$

therefore,

$$\int_{Q_{r/2}^z} v^p dx \leq ck^{-p}r^p \int_{Q_{r/2}^z} |Dv|^p dx.$$

Since $B_{r/4}^z \subset Q_{r/2}^z \subset B_{r/2}^z$, the last inequality allows us to assert that

$$\int_{B_{r/4}^z} v^p dx \leq ck^{-p}r^p \int_{B_{r/2}^z} |Dv|^p dx.$$

Combining this with (4.20), we obtain

$$\int_{B_{r/4}^z} v^p dx \leq ck^{-p} \int_{B_r^z} v^p dx.$$

At the same time, from Lemma 4.5, it follows that

$$\text{ess sup}_{B_{r/8}^z} v \leq cr^{-n/p} \left(\int_{B_{r/4}^z} v^p dx \right)^{1/p}.$$

Thus, we have

$$\text{ess sup}_{B_{r/8}^z} v \leq \frac{cr^{-n/p}}{k} \left(\int_{B_r^z} v^p dx \right)^{1/p} \leq \frac{c}{k} \text{ess sup}_{B_r^z} v.$$

To complete the proof, it remains to note that $k \rightarrow \infty$ as $\text{mes } \omega \rightarrow 0$. \square

Lemma 4.7. *Let $v \in W_p^1(B_r^z) \cap L_\infty(B_r^z)$, $r > 0$, $z \in \mathbb{R}^n$, be a non-negative solution of the inequality*

$$\text{div } A(x, Dv) + b(x)|Dv|^\alpha \geq \chi_\omega(x)h \quad \text{in } B_r^z \quad (4.21)$$

satisfying condition (4.14), where $h \geq 0$ is a real number, $\omega = \{x \in B_r^z : v(x) > 0\}$, and χ_ω is the characteristic function of ω . If $\text{mes } \omega > 0$ and, moreover,

$$\lim_{\xi \rightarrow +0} \text{ess sup}_{B_\xi^z} v \geq \gamma \text{ess sup}_{B_r^z} v \quad (4.22)$$

for some real number $\gamma > 0$, then

$$\text{ess sup}_{B_r^z} v \geq Cr^{p/(p-1)}h^{1/(p-1)},$$

where the constant $C > 0$ depends only on $n, p, \alpha, \gamma, C_1$, and C_2 .

Proof. We put $\varphi(x) = \varphi_0(|x-z|/r)$, where $\varphi_0 \in C^\infty(\mathbb{R})$ is a non-increasing function such that $\varphi_0|_{(-\infty, 1/4]} = 1$ and $\varphi_0|_{[1/2, \infty)} = 0$. Also let m and μ be the real numbers defined by (4.16). It is clear that $m > 0$ since $\text{mes } \omega > 0$.

Taking into account (4.21), we obtain

$$- \int_{B_r^z} A(x, Dv) D\varphi dx + \int_{B_r^z} b(x)|Dv|^\alpha \varphi dx \geq \int_{B_r^z} \chi_\omega(x)h\varphi dx,$$

whence it follows that

$$\int_{B_r^z} |A(x, Dv)| |D\varphi| dx + \int_{B_r^z} b(x) |Dv|^\alpha \varphi dx \geq h \operatorname{mes}\{\omega \cap B_{r/4}^z\}.$$

On the other hand,

$$\operatorname{mes}\{\omega \cap B_{r/4}^z\} \geq cr^n$$

in accordance with Lemma 4.6 and relation (4.22). Therefore, one can claim that

$$\int_{B_r^z} |A(x, Dv)| |D\varphi| dx + \int_{B_r^z} b(x) |Dv|^\alpha \varphi dx \geq cr^n h. \quad (4.23)$$

Using the Hölder inequality, we have

$$\begin{aligned} \int_{B_r^z} |A(x, Dv)| |D\varphi| dx &\leq C_2 \int_{B_{r/2}^z} |Dv|^{p-1} |D\varphi| dx \\ &\leq C_2 \left(\int_{B_{r/2}^z} |Dv|^p dx \right)^{(p-1)/p} \left(\int_{B_{r/2}^z} |D\varphi|^p dx \right)^{1/p} \\ &\leq cr^{(n-p)/p} \left(\int_{B_{r/2}^z} |Dv|^p dx \right)^{(p-1)/p}. \end{aligned}$$

At the same time,

$$\int_{B_r^z} b(x) |Dv|^\alpha \varphi dx \leq cr^{n(p-\alpha)/p} \mu \left(\int_{B_{r/2}^z} |Dv|^p dx \right)^{\alpha/p}.$$

Really, the last estimate is obvious for $\alpha = p$, whereas in the case of $\alpha < p$, it follows from the Hölder inequality

$$\int_{B_r^z} b(x) |Dv|^\alpha \varphi dx \leq \mu \left(\int_{B_{r/2}^z} |Dv|^p dx \right)^{\alpha/p} \left(\int_{B_{r/2}^z} \varphi^{p/(p-\alpha)} dx \right)^{(p-\alpha)/p}.$$

Hence, formula (4.23) allows us to assert that

$$r^{(n-p)/p} \left(\int_{B_{r/2}^z} |Dv|^p dx \right)^{(p-1)/p} + r^{n(p-\alpha)/p} \mu \left(\int_{B_{r/2}^z} |Dv|^p dx \right)^{\alpha/p} \geq cr^n h.$$

Since

$$\int_{B_{r/2}^z} |Dv|^p dx \leq cr^{-p} \int_{B_r^z} v^p dx \leq cr^{n-p} m^p$$

by Lemma 4.4, this implies the estimate

$$m^{p-1} + r^{p-\alpha} m^\alpha \mu \geq cr^p h,$$

whence in accordance with the condition

$$r^{p-\alpha} m^\alpha \mu \leq \frac{C_1}{2} m^{p-1}$$

which follows from (4.14) we obtain

$$m^{p-1} \geq cr^p h.$$

The proof is completed. \square

Lemma 4.8. *Let $v \in W_p^1(B_{r_1, r_2}) \cap L_\infty(B_{r_1, r_2})$ be a non-negative solution of the inequality*

$$\operatorname{div} A(x, Dv) + b(x)|Dv|^\alpha \geq \chi_\omega(x)h \quad \text{in } B_{r_1, r_2}, \quad (4.24)$$

where $0 < r_1 < r_2$ and $h \geq 0$ are real numbers, $\omega = \{x \in B_{r_1, r_2} : v(x) > 0\}$, and χ_ω is the characteristic function of ω . If there exists a real number $m > 0$ such that $M(r; v) = m$ for all $r \in (r_1, r_2)$, then $h = 0$.

Proof. We put $r_* = (r_1 + r_2)/2$. There is a point $z \in S_{r_*}$ such that

$$\lim_{\xi \rightarrow +0} \operatorname{ess\,sup}_{B_\xi^z} v = m.$$

Take real numbers $0 < \varepsilon < m$ and $0 < r < (r_2 - r_1)/2$ satisfying the relation

$$\varepsilon^{\alpha-p+1} r^{p-\alpha} \operatorname{ess\,sup}_{B_{r_1, r_2}} b \leq \frac{C_1}{4}.$$

We denote

$$\tilde{v}(x) = \max\{v(x) - m + \varepsilon, 0\}.$$

By Corollary 4.1, \tilde{v} is a non-negative solution of the inequality

$$\operatorname{div} A(x, D\tilde{v}) + b(x)|D\tilde{v}|^\alpha \geq \chi_{\tilde{\omega}}(x)h \quad \text{in } B_r^z, \quad (4.25)$$

where $\tilde{\omega} = \{x \in B_r^z : \tilde{v}(x) > 0\} \subset \omega$. Thus,

$$\varepsilon \geq cr^{p/(p-1)} h^{1/(p-1)}$$

by Lemma 4.7. Passing in the last expression to the limit as $\varepsilon \rightarrow +0$, we complete the proof. \square

Lemma 4.9. *Let $v \in W_p^1(B_{r_1, r_2}) \cap L_\infty(B_{r_1, r_2})$ be a non-negative solution of (4.24) such that $M(\cdot; v)$ is a non-decreasing function on $[r_1, r_2]$, $M(r_2; v) > M(r_1; v) \geq \gamma M(r_2; v)$ and, moreover,*

$$(M(r_2; v) - M(r_1; v))^{\alpha-p+1} (r_2 - r_1)^{p-\alpha} \operatorname{ess\,sup}_{B_{r_1, r_2}} b \leq \frac{C_1}{4}, \quad (4.26)$$

where $0 < r_1 < r_2$, $0 < \gamma < 1$, and $h \geq 0$ are real numbers, $\omega = \{x \in B_{r_1, r_2} : v(x) > 0\}$, and χ_ω is the characteristic function of ω . Then

$$(M(r_2; v) - M(r_1; v))^{p-1} \geq C(r_2 - r_1)^p h, \quad (4.27)$$

where the constant $C > 0$ depends only on $n, p, \alpha, \gamma, C_1$, and C_2 .

Proof. We denote $r = (r_2 - r_1)/2$ and $r_* = (r_1 + r_2)/2$. Take a point $z \in S_{r_*}$ such that

$$\lim_{\xi \rightarrow +0} \operatorname{ess\,sup}_{B_\xi^z} v \geq M(r_*; v).$$

By the maximum principle, we have

$$M(r_2; v) = \operatorname{ess\,sup}_{B_{r_1, r_2}} v \geq \operatorname{ess\,sup}_{B_r^z} v.$$

It can also be seen that

$$M(r_*; v) \geq M(r_1; v) \geq \gamma M(r_2; v).$$

If $M(r_2; v) = M(r_*; v)$, then (4.27) is obvious since $h = 0$ by Lemma 4.8. Hence, one can assume that $M(r_2; v) > M(r_*; v)$. In the case of $M(r_2; v) \geq 2M(r_*; v)$, we obtain

$$\operatorname{ess\,sup}_{B_r^z} v \leq M(r_2; v) \leq 2(M(r_2; v) - M(r_*; v));$$

therefore,

$$\begin{aligned} r^{p-\alpha} \left(\operatorname{ess\,sup}_{B_r^z} v \right)^{\alpha-p+1} \operatorname{ess\,sup}_{B_r^z} b &\leq 2^{1+2(\alpha-p)} (M(r_2; v) - M(r_*; v))^{\alpha-p+1} \\ &\quad \times (r_2 - r_1)^{p-\alpha} \operatorname{ess\,sup}_{B_{r_1, r_2}} b \\ &\leq \frac{C_1}{2} \end{aligned}$$

in accordance with (4.26). Thus, Lemma 4.7 allows us to assert that

$$M(r_2; v) - M(r_*; v) \geq \frac{1}{2} \operatorname{ess\,sup}_{B_r^z} v \geq cr^{p/(p-1)} h^{1/(p-1)},$$

whence (4.27) follows at once.

Now, let $M(r_2; v) < 2M(r_*; v)$. We put

$$\tilde{v}(x) = \max\{v(x) + M(r_2; v) - 2M(r_*; v), 0\}.$$

By Corollary 4.1, the function \tilde{v} is a non-negative solution of inequality (4.25), where $\tilde{\omega} = \{x \in B_r^z : \tilde{v}(x) > 0\} \subset \omega$ as before. In addition, we have

$$\lim_{\xi \rightarrow +0} \operatorname{ess\,sup}_{B_\xi^z} \tilde{v} \geq M(r_2; v) - M(r_*; v).$$

and

$$\operatorname{ess\,sup}_{B_r^z} \tilde{v} \leq \operatorname{ess\,sup}_{B_{r_1, r_2}} \tilde{v} = 2(M(r_2; v) - M(r_*; v)).$$

Thus, (4.26) implies the estimate

$$\begin{aligned} r^{p-\alpha} \left(\operatorname{ess\,sup}_{B_r^z} \tilde{v} \right)^{\alpha-p+1} \operatorname{ess\,sup}_{B_r^z} b &\leq 2^{1+2(\alpha-p)} (M(r_2; v) - M(r_*; v))^{\alpha-p+1} \\ &\quad \times (r_2 - r_1)^{p-\alpha} \operatorname{ess\,sup}_{B_{r_1, r_2}} b \\ &\leq \frac{C_1}{2}. \end{aligned}$$

To complete the proof, it remains to note that

$$M(r_2; v) - M(r_*; v) \geq \frac{1}{2} \operatorname{ess\,sup}_{B_r^z} \tilde{v} \geq cr^{p/(p-1)} h^{1/(p-1)}$$

according to Lemma 4.7. □

Lemma 4.10. *Let $\eta : (r_1, r_2) \rightarrow [0, \infty)$ be a non-decreasing function such that $\eta(r_1 + 0) = 0$, $\eta(r_2 - 0) > \varepsilon$ and, moreover, $\eta(r - 0) = \eta(r)$ for all $r \in (r_1, r_2)$, where $r_1 < r_2$ and $\varepsilon > 0$ are real numbers. Then there is a real number $\xi \in (r_1, r_2)$ satisfying the conditions $\eta(\xi) \leq \varepsilon$ and $\eta(\xi + 0) \geq \varepsilon$.*

Proof. Consider sequences of real numbers $\{\mu_i\}_{i=1}^\infty$ and $\{m_i\}_{i=1}^\infty$ constructed by induction. We put $\mu_1 = r_1$ and $m_1 = r_2$. Assume further that μ_i and m_i are already known. If $\eta((\mu_i + m_i)/2) > \varepsilon$, then we take $\mu_{i+1} = \mu_i$ and $m_{i+1} = (\mu_i + m_i)/2$. Otherwise we take $\mu_{i+1} = (\mu_i + m_i)/2$ and $m_{i+1} = m_i$.

It can easily be seen that $0 < m_i - \mu_i \leq 2^{1-i}(r_2 - r_1)$ and $[\mu_{i+1}, m_{i+1}] \subset [\mu_i, m_i]$ for all $i = 1, 2, \dots$. Therefore,

$$\lim_{i \rightarrow \infty} \mu_i = \lim_{i \rightarrow \infty} m_i = \xi$$

for some $\xi \in [r_1, r_2]$. If $\xi = r_1$, then

$$\eta(r_1 + 0) = \lim_{i \rightarrow \infty} \eta(m_i) \geq \varepsilon.$$

This contradicts the fact that $\eta(r_1 + 0) = 0$. On the other hand, if $\xi = r_2$, then

$$\eta(r_2 - 0) = \lim_{i \rightarrow \infty} \eta(\mu_i) \leq \varepsilon$$

and we derive a contradiction once more. Thus, one can claim that $\xi \in (r_1, r_2)$. In so doing, the relations $\eta(\xi) \leq \varepsilon$ and $\eta(\xi + 0) \geq \varepsilon$ follow directly from the definition of the sequences $\{\mu_i\}_{i=1}^\infty$ and $\{m_i\}_{i=1}^\infty$.

The proof is completed. \square

Lemma 4.11. *In the conditions of Lemma 4.9, let the inequality*

$$(M(r_2; v) - M(r_1; v))^{\alpha-p+1} (r_2 - r_1)^{p-\alpha} \operatorname{ess\,sup}_{B_{r_1, r_2}} b \geq \lambda \quad (4.28)$$

be valid instead of (4.26) for some real number $\lambda > 0$. Then

$$(M(r_2; v) - M(r_1; v))^\alpha \operatorname{ess\,sup}_{B_{r_1, r_2}} b \geq C(r_2 - r_1)^\alpha h, \quad (4.29)$$

where the constant $C > 0$ depends only on $n, p, \alpha, \gamma, \lambda, C_1$, and C_2 .

Proof. It can be assumed that $M(\cdot; v)$ is a strictly increasing function on $[r_1, r_2]$; otherwise $h = 0$ by Lemma 4.8 and (4.29) is evident. We denote

$$\psi(r) = (M(r_2; v) - M(r; v))^{\alpha-p+1} (r_2 - r)^{p-\alpha} \mu,$$

where

$$\mu = \operatorname{ess\,sup}_{B_{r_1, r_2}} b.$$

Also let $\varepsilon = \min\{\lambda, C_1/4\}$ and $r_* = \inf\{r \in (r_1, r_2) : \psi(r) \leq \varepsilon\}$.

By Lemma 4.9, we have

$$(M(r_2; v) - M(r_* + 0; v))^{p-1} \geq c(r_2 - r_*)^p h.$$

Multiplying this by the inequality

$$(M(r_2; v) - M(r_*; v))^{\alpha-p+1} (r_2 - r_*)^{p-\alpha} \mu \geq \varepsilon$$

which follows from (4.28) if $r_* = r_1$ or from the relation $\psi(r_*) = \psi(r_* - 0)$ if $r_* \in (r_1, r_2)$, we obtain

$$(M(r_2; v) - M(r_*; v)) \mu^{1/\alpha} \geq c(r_2 - r_*) h^{1/\alpha}. \quad (4.30)$$

In the case of $r_* = r_1$, we complete the proof. Thus, it can be assumed that $r_* > r_1$. Let us construct a finite sequence of real numbers $\{\xi_i\}_{i=1}^k$. We put $\xi_1 = r_1$. Assume further that ξ_i is already known. If

$$(M(r_2; v) - M(\xi_i + 0; v))^{\alpha-p+1} (r_2 - \xi_i)^{p-\alpha} \mu \leq \varepsilon,$$

then we put $k = i$ and stop. Otherwise we take $\xi_{i+1} \in (\xi_i, r_2)$ such that

$$(M(\xi_{i+1}; v) - M(\xi_i + 0; v))^{\alpha-p+1}(\xi_{i+1} - \xi_i)^{p-\alpha}\mu \leq \varepsilon$$

and

$$(M(\xi_{i+1} + 0; v) - M(\xi_i + 0; v))^{\alpha-p+1}(\xi_{i+1} - \xi_i)^{p-\alpha}\mu \geq \varepsilon. \quad (4.31)$$

By Lemma 4.10, such a real number ξ_{i+1} obviously exists. The above procedure must terminate at a finite step; otherwise, according to (4.31), we have

$$\begin{aligned} M(\xi_{i+1} + 0; v) - M(\xi_i + 0; v) &\geq \left(\frac{\varepsilon}{\mu(\xi_{i+1} - \xi_i)^{p-\alpha}} \right)^{1/(\alpha-p+1)} \\ &\geq \left(\frac{\varepsilon}{\mu(r_2 - r_1)^{p-\alpha}} \right)^{1/(\alpha-p+1)}, \quad i = 1, 2, \dots, \end{aligned}$$

whence it follows that

$$M(r_2; v) - M(r_1; v) \geq \sum_{i=1}^{\infty} (M(\xi_{i+1} + 0; v) - M(\xi_i + 0; v)) = \infty.$$

By Lemma 4.9,

$$(M(\xi_{i+1}; v) - M(\xi_i + 0; v))^{p-1} \geq c(\xi_{i+1} - \xi_i)^p h, \quad i = 1, \dots, k-1.$$

Multiplying this by (4.31), we obtain

$$(M(\xi_{i+1} + 0; v) - M(\xi_i + 0; v))\mu^{1/\alpha} \geq c(\xi_{i+1} - \xi_i)h^{1/\alpha}, \quad i = 1, \dots, k-1.$$

Therefore,

$$(M(\xi_k + 0; v) - M(r_1 + 0; v))\mu^{1/\alpha} \geq c(\xi_k - r_1)h^{1/\alpha}.$$

Since $\xi_k \geq r_*$, the last estimate and (4.30) allow us to assert that

$$(M(r_2; v) - M(r_1 + 0; v))\mu^{1/\alpha} \geq c(r_2 - r_1)h^{1/\alpha}.$$

The proof is completed. \square

Proof of Lemmas 3.1–3.3. Corollary 4.2 implies that $M(\cdot; u)$ is a non-decreasing function on $[r_1, r_2]$. We put $\omega = \{x \in \Omega_{r_1, r_2} : u(x) > \beta M(r_1; u)\}$,

$$v(x) = \begin{cases} u(x) - \beta M(r_1; u), & x \in \omega, \\ 0, & x \in \Omega_{r_1, r_2} \setminus \omega \end{cases}$$

and

$$h = \operatorname{ess\,inf}_{\Omega_{r_1, r_2}} q \inf_{(\beta M(r_1; u), M(r_2; u))} g.$$

In so doing, for Lemma 3.1, we take $\beta = 1/2$. By Lemma 4.1, v is a non-negative solution of inequality (4.24). Thus, to complete the proof, it remains to apply Lemmas 4.8, 4.9, and 4.11, respectively. \square

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